

The Role of Conservation Principles in the Abraham–Minkowski Controversy

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(Dated: April 8, 2016)

The Abraham–Minkowski controversy refers to a long-standing inability to adequately address certain issues involving the conservation of the momentum of an electromagnetic field in a linear dielectric medium. We apply the usual assumption of a material subsystem that couples to the electromagnetic subsystem such that the total energy and total momentum are conserved. We then construct the total energy–momentum tensor from the total energy density and the total momentum density. Applying conservation principles to the total energy–momentum tensor, we construct the tensor energy–momentum continuity equation. We show that one of the components of the tensor energy–momentum continuity equation, the energy continuity equation, is manifestly false. We conclude that the Abraham–Minkowski controversy is unresolvable because the extant principles of conservation are inconsistent in a simple linear dielectric medium.

I. INTRODUCTION

The century-long history of the Abraham–Minkowski controversy [1–14] is a search for some provable description of momentum and momentum conservation for electromagnetic fields in dielectric media. A wide variety of physical principles have been applied to establish the priority of one type of momentum over another, or to establish that the Abraham and Minkowski formulations are equally valid. Typically, one assumes some fundamental physical principle or law and the correctness of the results are affirmed by the fundamental nature of the laws that are used as the basis of the analyses, such as the macroscopic Maxwell equations, the Lorentz dipole force, symmetrized Minkowski tensor, the constancy of the center-of-mass energy velocity, Lorentz invariance, or spatially averaged microscopic fields. In order to satisfy global conservation principles, an appropriate material momentum must accompany these electromagnetic momentums such that the sum of the electromagnetic and material momentums is the total momentum that is conserved in a closed system.

For a thermodynamically closed system consisting of a propagating light field and an antireflection-coated simple linear dielectric, the total momentum [3, 12–14],

$$\mathbf{G}_T = \int_{\sigma} \mathbf{g}_T dv = \int_{\sigma} \frac{n\mathbf{E} \times \mathbf{B}}{c} dv, \quad (1.1)$$

is temporally invariant when the total momentum density \mathbf{g}_T is integrated over all-space σ . Global conservation of the total momentum \mathbf{G}_T , Eq. (1.1), has been documented for an antireflection-coated block of a simple linear dielectric material situated in free-space that is illuminated by a quasimonochromatic field [13, 14]. The modern resolution of the Abraham–Minkowski momentum controversy is to adopt a scientific conformity in which the Minkowski

momentum,

$$\mathbf{G}_M = \int_{\sigma} \mathbf{g}_M dv = \int_{\sigma} \frac{\mathbf{D} \times \mathbf{B}}{c} dv, \quad (1.2)$$

and the Abraham momentum,

$$\mathbf{G}_A = \int_{\sigma} \mathbf{g}_A dv = \int_{\sigma} \frac{\mathbf{E} \times \mathbf{H}}{c} dv, \quad (1.3)$$

are both “correct” forms of electromagnetic momentum with the understanding that neither is the total momentum [3, 4, 7, 11]. Then, either the Minkowski momentum or the Abraham momentum can be adopted as the momentum of the electromagnetic field as long as that momentum is accompanied by the appropriate material momentum, $\mathbf{G}_T - \mathbf{G}_A$ or $\mathbf{G}_T - \mathbf{G}_M$. In the most general interpretation of the consensus resolution of the Abraham–Minkowski controversy, only the total momentum has physical meaning and the total momentum can be divided into arbitrary field and material components [3].

It is well known that a material momentum accompanies the electromagnetic momentum in a dielectric and that their sum, the total momentum, is conserved. The components of the total momentum density, along with the total energy density, are elements of the total energy–momentum tensor. In this article we report that the tensor total energy–momentum continuity equation that is constructed from the total energy–momentum tensor is false. The Abraham–Minkowski controversy cannot be resolved using the existing theory because the principles of conservation are inconsistent in a linear dielectric medium.

II. ENERGY–MOMENTUM TENSORS

The theory of continuum electrodynamics is based on the macroscopic Maxwell equations

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0 \quad (2.1a)$$

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$$\nabla \cdot \mathbf{B} = 0 \quad (2.1b)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (2.1c)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (2.1d)$$

and the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} \quad (2.2a)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (2.2b)$$

for an absorptionless linear medium. For clarity and concision, we treat a limiting case in which the pulse is sufficiently monochromatic and the center frequency of the exciting field is sufficiently far from material resonances that dispersion can be treated parametrically and otherwise ignored.

The energy and momentum continuity equations are the counterparts of conservation laws when the system consists of a continuous flow rather than localized and enumerated discrete particles. In the particular case of macroscopic electromagnetic energy and momentum, the continuity equations are easily derived from the macroscopic Maxwell equations, Eqs. (2.1)–(2.2). We subtract the scalar product of Eq. (2.1a) with \mathbf{E} from the scalar product of Eq. (2.1c) with \mathbf{H} and apply a common vector identity to produce

$$\frac{1}{c} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0. \quad (2.3)$$

We define the macroscopic electromagnetic energy density

$$\rho_e = \frac{1}{2} (\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2) \quad (2.4)$$

and the Poynting energy flux vector

$$\mathbf{S} = c \mathbf{E} \times \mathbf{H}, \quad (2.5)$$

as usual, to obtain Poynting's theorem

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{S} = 0 \quad (2.6)$$

for continuity of electromagnetic energy in a linear medium. We can also form

$$\frac{\partial}{\partial t} \frac{\mathbf{D} \times \mathbf{B}}{c} = -\mathbf{D} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{H}) \quad (2.7)$$

from the difference of cross products of Eqs. (2.1a) and (2.1c) with macroscopic fields. In what follows, the index convention for Greek letters is that they belong to $\{0, 1, 2, 3\}$ and lower case Roman indices from the middle of the alphabet are in $\{1, 2, 3\}$. Defining the stress-tensor

$$W_{ij} = -E_i D_j - H_i B_j + \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \delta_{ij} \quad (2.8)$$

and the Minkowski momentum density

$$\mathbf{g}_M = \frac{\mathbf{D} \times \mathbf{B}}{c} \quad (2.9)$$

yields the momentum continuity equation

$$\frac{\partial \mathbf{g}_M}{\partial t} + \nabla \cdot \mathbf{W} = -(\nabla \varepsilon) \frac{\mathbf{E}^2}{2} - (\nabla \mu) \frac{\mathbf{H}^2}{2}. \quad (2.10)$$

As a matter of linear algebra, we can write the energy continuity equation, Eq. (2.6), and the three scalar equations from the momentum continuity equation, Eq. (2.10), as a single matrix continuity equation

$$\partial_\beta T_M^{\alpha\beta} = f_M^\alpha, \quad (2.11)$$

where

$$\partial_\beta = \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (2.12)$$

is the four-divergence operator,

$$\mathbf{f}_M = -\frac{\nabla \varepsilon}{\varepsilon} \frac{\mathbf{E} \cdot \mathbf{D}}{2} - \frac{\nabla \mu}{\mu} \frac{\mathbf{H} \cdot \mathbf{B}}{2} \quad (2.13)$$

is the Minkowski force density that is a source, or sink, of electromagnetic momentum for the field. (The force density on the dielectric is the Helmholtz force density $\mathbf{f}_H = -\mathbf{f}_M$.) Also, $f_M^\alpha = (0, \mathbf{f}_M)$ is the Minkowski four-force density, and

$$T_M^{\alpha\beta} = \begin{bmatrix} \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) & (\mathbf{E} \times \mathbf{H})_1 & (\mathbf{E} \times \mathbf{H})_2 & (\mathbf{E} \times \mathbf{H})_3 \\ (\mathbf{D} \times \mathbf{B})_1 & W_{11} & W_{12} & W_{13} \\ (\mathbf{D} \times \mathbf{B})_2 & W_{21} & W_{22} & W_{23} \\ (\mathbf{D} \times \mathbf{B})_3 & W_{31} & W_{32} & W_{33} \end{bmatrix} \quad (2.14)$$

is, by construction, a four-by-four matrix. The Minkowski matrix differential continuity equation, Eq. (2.11), has the outward appearance of being a tensor energy-momentum continuity equation and it was assumed to be so. Then, the matrix $T_M^{\alpha\beta}$, Eq. (2.14), became known as the Minkowski energy-momentum tensor.

The origin story of the Abraham-Minkowski controversy is that the Minkowski energy-momentum tensor, Eq. (2.14), is not diagonally symmetric. Motivated by the need to preserve the principle of conservation of angular momentum, Abraham symmetrized the energy-momentum tensor by re-defining the linear momentum density,

$$\mathbf{g}_A = \mathbf{E} \times \mathbf{H}/c, \quad (2.15)$$

to be proportional to the Poynting vector. Substituting the Abraham momentum density into Eq. (2.10), we can construct the matrix continuity equation

$$\partial_\beta T_A^{\alpha\beta} = f_A^\alpha, \quad (2.16)$$

where

$$f_A^\alpha = \left(0, -\frac{\nabla \varepsilon}{\varepsilon} \frac{\mathbf{E} \cdot \mathbf{D}}{2} - \frac{\nabla \mu}{\mu} \frac{\mathbf{H} \cdot \mathbf{B}}{2} + \frac{\partial}{\partial t} \frac{(1 - n^2) \mathbf{E} \times \mathbf{H}}{c} \right) \quad (2.17)$$

is the Abraham four-force,

$$T_A^{\alpha\beta} = \begin{bmatrix} \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) & (\mathbf{E} \times \mathbf{H})_1 & (\mathbf{E} \times \mathbf{H})_2 & (\mathbf{E} \times \mathbf{H})_3 \\ (\mathbf{E} \times \mathbf{H})_1 & W_{11} & W_{12} & W_{13} \\ (\mathbf{E} \times \mathbf{H})_2 & W_{21} & W_{22} & W_{23} \\ (\mathbf{E} \times \mathbf{H})_3 & W_{31} & W_{32} & W_{33} \end{bmatrix} \quad (2.18)$$

is known as the Abraham energy-momentum tensor, and ∂_β is again the four-divergence operator.

III. CONSERVATION

It has been established in the scientific literature that neither the Abraham momentum nor the Minkowski momentum is conserved in an isolated system [3]. Although the definitions of the linear momentum and the energy-momentum tensor have been the most contentious issues, the main challenge of the Abraham-Minkowski controversy is to construct a tensor energy-momentum continuity equation. The conditions [15]

$$\partial_\beta T^{\alpha\beta} = 0 \quad (3.1a)$$

$$T^{\alpha\beta} = T^{\beta\alpha} \quad (3.1b)$$

$$\int_\sigma T^{\alpha 0} dv \text{ is temporally invariant} \quad (3.1c)$$

insure that the energy and momentum conservation laws are satisfied for an unimpeded continuous flow of electromagnetic radiation. Due to the arbitrary size of the dielectric, we can invoke the limit that the gradient of the material properties n , ε , and μ are sufficiently small that source terms involving these gradients can be neglected. In this limit, Eq. (2.11) becomes

$$\partial_\beta T_M^{\alpha\beta} = 0 \quad (3.2)$$

and has the appearance of the tensor energy-momentum continuity equation in the form of Eq. (3.1a), where $T^{\alpha\beta} = T_M^{\alpha\beta}$. In fact, Eq. (3.2) proves that the Minkowski momentum, Eq. (1.2), is conserved. This result is contradicted by Eq. (3.1c) because global conservation principles prove that the Minkowski momentum is not conserved for the thermodynamically closed system that is being treated here [3, 12–14].

There exists an accepted procedure to remove the contradiction in which the energy-momentum tensor of a material subsystem is added to the energy-momentum tensor of an electromagnetic subsystem [3–12]. The total energy-momentum tensor is

$$T^{\alpha\beta} = T_{em}^{\alpha\beta} + T_{matter}^{\alpha\beta}. \quad (3.3)$$

The electromagnetic subsystem energy-momentum tensor is typically the Abraham tensor or the Minkowski tensor. Because the total energy-momentum tensor is

unique, the selection of an electromagnetic tensor determines the material tensor. By definition, the total energy and the total linear momentum are unique conserved quantities that satisfy the condition Eq. (3.1c). Based on global conservation principles, we can take the total energy density

$$T^{00} = T_{em}^{00} + T_{matter}^{00} = \rho_e = (n^2 \mathbf{E}^2 + \mathbf{B}^2)/2 \quad (3.4)$$

and the components of the total momentum density

$$T^{i0} = T_{em}^{i0} + T_{matter}^{i0} = \mathbf{g}_{Ti} = (n\mathbf{E} \times \mathbf{B}/c)_i \quad (3.5)$$

as given quantities. Applying the total energy density, Eq. (3.4), the total momentum density, Eq. (3.5), and conservation of angular momentum, Eq. (3.1b), one constructs the total energy-momentum tensor

$$T^{\alpha\beta} = \begin{bmatrix} \frac{1}{2}(n^2 \mathbf{E}^2 + \mathbf{B}^2) & (n\mathbf{E} \times \mathbf{B})_1 & (n\mathbf{E} \times \mathbf{B})_2 & (n\mathbf{E} \times \mathbf{B})_3 \\ (n\mathbf{E} \times \mathbf{B})_1 & Y_{11} & Y_{12} & Y_{13} \\ (n\mathbf{E} \times \mathbf{B})_2 & Y_{21} & Y_{22} & Y_{23} \\ (n\mathbf{E} \times \mathbf{B})_3 & Y_{31} & Y_{32} & Y_{33} \end{bmatrix}. \quad (3.6)$$

The components of Y are left unspecified as they are not needed for the present discussion. Substituting the total energy-momentum tensor, Eq. (3.6), into the continuity equation, Eq. (3.1a), produces

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (n\mathbf{S}) = 0 \quad (3.7)$$

for the $\alpha = 0$ component. The energy continuity equation, Eq. (3.7), is manifestly false because the two terms of Eq. (3.7) depend on different powers of the parameter n . This is obvious if we compare Eq. (3.7) with Poynting's theorem, Eq. (2.6), and it can also be proved by evaluating the amplitudes of the components. Then the conservation principle embodied in the continuity equation, Eq. (3.1a), is inconsistent with conservation of total energy and conservation of total linear momentum, Eq. (3.1c), conservation of total angular momentum, Eq. (3.1b), and with the total energy-momentum tensor, Eq. (3.6).

IV. DISCUSSION

The scientific literature contains a large number of theoretical and experimental studies that claim to resolve the Abraham-Minkowski controversy. Here, we discuss how the results of the previous section relate to examples from the prior work. In particular, we find that the prior work stopped short of considering the validity of the tensor total energy-momentum continuity equation.

In an influential 1973 article, Gordon [12] uses a microscopic model of the dielectric in terms of electric dipoles.

Assuming a dilute vapor in which the dipoles do not interact with each other or their environment, Gordon averages the microscopic Lorentz dipole force on a particle with linear polarizability α

$$\mathbf{f} = \alpha \left((\mathbf{E} \cdot \nabla) \mathbf{E} + \frac{d\mathbf{E}}{dt} \times \mathbf{B} \right) \quad (4.1)$$

to obtain a macroscopic Lorentz force density that is integrated to produce the material momentum density

$$\mathbf{g}_{matter} = (n-1) \frac{\mathbf{E} \times \mathbf{B}}{c}. \quad (4.2)$$

Gordon added the material momentum density to the Abraham momentum density for the electromagnetic field to obtain $\mathbf{g}_T = (n/c) \mathbf{E} \times \mathbf{B}$ for the total momentum density. The corresponding total momentum, Eq. (1.1), has been proved to be conserved in the situations that are considered here. This result, alone, is insufficient to construct the tensor total energy–momentum continuity equation. If we also require that the total angular momentum and the total energy are conserved, then we can construct the total energy–momentum tensor, Eq. (3.6). It was shown in the preceding section that the resulting total energy continuity equation, Eq. (3.7), is false.

In similarly influential work, Barnett [4] and Barnett and Loudon [11] prove that the Abraham momentum is the kinetic momentum and the Minkowski momentum is the canonical momentum. They assert that both results are correct because 1) they were derived from fundamental principles and 2) the total momentum can be constructed by adding a material momentum, different in each case, to the Abraham kinetic momentum and to the Minkowski canonical momentum. The sum of the kinetic momentum and its material momentum is equal to the sum of the canonical momentum and its material momentum. In each case, the sum is required to be the total linear momentum, which is conserved. Then, Eq. (1.1) is the total linear momentum and we can construct the total energy–momentum tensor, Eq. (3.6), as before. Applying the four-divergence operator to the total energy–momentum tensor as shown in Eq. (3.1a), the component of the energy–momentum continuity equation that relates to continuity of energy, Eq. (3.7), remains false.

In a microscopic approach, the material momentum is modeled as the kinematic momentum of individual

particles of matter [3, 9]. The total energy–momentum tensor is assumed to be the sum of an electromagnetic tensor and the dust tensor. However, the dust energy–momentum tensor is usually applied to a thermodynamically closed system consisting of non-interacting neutral particles in an incoherent unimpeded flow. Then we must derive the relationships between the electromagnetic and material subsystems from the conservation principles, Eqs. (3.1b) and (3.1c), for the composite total energy–momentum tensor, Eq. (3.3). The resulting total energy–momentum tensor is unique and this tensor, Eq. (3.6), leads to provably false conservation principles, as before.

A number of experiments have been performed in order to determine the momentum of the field in a dielectric, notably Refs. [16–18]. The results of electromagnetic momentum experiments, either the Minkowski momentum or the Abraham momentum, are not sufficient to construct the total energy–momentum tensor or the tensor total energy momentum continuity equation. We must rely on conservation principles to construct the total energy–momentum tensor, Eq. (3.6), which is unique. Substituting the total energy–momentum tensor into Eq. (3.1a) results in a false statement of energy conservation, Eq. (3.7).

V. CONCLUSIONS

The Abraham–Minkowski controversy cannot be resolved using the existing theory because the principles of conservation are inconsistent in a linear dielectric medium. Specifically, the first row and first column of the total energy–momentum tensor continuity equations can be constructed by conservation of total energy, conservation of total linear momentum, and conservation of total angular momentum. However, the four-divergence of the total energy–momentum tensor produces a total energy continuity equation that is manifestly false. This result shows that more than the assumption of a material subsystem to add to an electromagnetic subsystem will be required to resolve the Abraham–Minkowski controversy. In future publications, we will show how to repair the conservation principles and related laws of physics in a dielectric medium.

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